Axisymmetric Plasma Equilibria with Incompressible Flow

Klaus Elsässer and Alfred Heimsoth
Institut für Theoretische Physik, Ruhr-Universität Bochum, FRG

Z. Naturforsch. **41 a,** 883–896 (1986); received February 21, 1986

The reduction of the ideal MHD equations to a scalar elliptic equation ("ξ-equation") is discussed for an axisymmetric equilibrium with an incompressible subcritical flow field. First it is shown that the most general equilibrium of this type can be obtained by a numerical solution for the case of a purely toroidal motion ("isorotation"); it is defined by prescribing three flux functions. Then a gauge transformation $\xi(\Psi)$ corresponding to a re-labeling of the flux surfaces (which are kept fixed) allows one to introduce a finite subcritical poloidal flow which minimizes the total energy content. Finally a nonlinear analytical model problem is used to test the efficiency of a numerical procedure for solving the ξ-equation; fast convergence and good results are obtained with a Newton-type iteration which has been discretized by the finite element method.

1. Introduction

In a high-temperature plasma two widely separated time scales are usually assumed to be important: One refers to the establishment of the force balance as described by the equations of ideal magnetohydrodynamics (MHD); the other and longer one is associated with dissipative processes like heat conduction, electric resistivity or viscous drag. A "quiet" toroidal plasma is, therefore, usually described by a static MHD-equilibrium; a small flow velocity may then be determined afterwards by considering a steady state with sources and sinks in the balance equations [1]. But nowadays the heating power (e.g. by neutral beams) and, consequently, the losses are so high that it is of interest to consider MHD equilibria including a stationary flow field. Recently a numerical scheme has been described and used to obtain such a stationary MHD equilibrium [2] assuming axisymmetry and adiabatic pressure variations. Such a treatment is not easy for several reasons: First, one has to solve the nonlinear partial differential equation for the poloidal magnetic flux function Ψ simultaneously with the algebraic equation for the plasma density (Bernoulli equation) which may convert the elliptic nature of the former into a hyperbolic nature and vice versa, depending on the particular parameter and space region of interest [3]. Secondly, the stationary equilibrium is specified by five "flux functions" (i.e. functions of Ψ) instead of two in the case of an axisymmetric static equilibrium, so it is more difficult to specify a solution which is both mathematically well-behaved and realizable by a physical experiment. It is, therefore, not probable that this type of treatment will become a standard procedure like the calculation of static equilibria. On the other hand, the confinement usually becomes worse with increasing heating power; a measured toroidal Mach number of 0.6 with respect to the local sound velocity [4] is already a high value, and it may turn out that a better strategy will keep the velocities smaller. It is then of interest to know some simple modifications of the usual codes for static equilibria in order to include modest velocities.

In analogy to hydrodynamics one could think of incompressible flows; they can be realized if the flow velocity is sufficiently smaller than the local sound velocity. In this case the poloidal Alfvén Mach number is smaller than $\beta_p^{1/2}$ where β_p is the poloidal plasma beta. This means that for present-day Tokamaks with $\beta_p \approx 0\,(1)$ the incompressible flow in the poloidal (or meridional) plane is also subcritical with respect to the local poloidal Alfvén velocity. Here we will concentrate the discussion on this case though situations with $\beta_p \gg 1$ – including astrophysical plasmas like the interior part of stars – may also be treated similarly. Like in the case of a static equilibrium (e.g. [5]–[7]) the basic equations for an axisymmetric stationary equilibrium have been derived and discussed several times before the

Reprint requests to Herrn Prof. K. Elsässer, Institut für Theoretische Physik, Ruhr-Universität Bochum, Postfach 10 21 48, 4630 Bochum.

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computational means of solving them where available. Chandrasekhar [8] and Woltjer [9] derived equations for the case of an incompressible flow; later on the equations for adiabatic flows have also been considered ([10]–[12]), and since that time several analytic solutions have been found ([13]–[15]). Moreover, the case of a purely toroidal flow (which is numerically more accessible) has also been solved with the computer in [16]. The number of free flux functions and the stability problem of such solutions has been discussed in [17]–[19].

But it seems that in the more modest problem of an incompressible axisymmetric stationary flow several inherent simplifications have not yet been observed; they are discussed in Sections 2–6. For the convenience of the reader we derive the scalar flux equation (A 36) in the appendix for the particular case where the temperature is assumed to be a flux function; this is in agreement with the usual reasoning and corresponds to adiabatic pressure variations with index $\gamma = 1$ if the equation of state for an ideal gas is used. In order to save the symmetry between magnetic and hydrodynamic flux functions, we follow Morozov and Solov'ev [11] and assume that all flux functions depend on a function ξ which is not further specified; it may be viewed as a gauge transformation in the sense that the flux surfaces are not changed (the physical quantities, however, are changed). With some effort it can be shown that our Eq. (A.36) is the same as the corresponding eq. (II) p. 283 of Solov'ev [12], and as a Ψ-equation it is also the same as eq. (7) of Hameiri [19] for $\gamma = 1$. Section 2 shows the constraints introduced by the incompressibility assumption; the most important one is the requirement that the plasma density ϱ_0 becomes a flux function. In Sect. 3 the ξ-equation is given for this case. It can be solved in principle by specifying only three flux functions within a particular gauge $\xi(\Psi)$ if gravitation is neglected (Section 4). These flux functions can be characterized as follows: Assume that a static equilibrium defined by the two flux functions $\hat{p}_0$ (pressure) and $\hat{I}_0$ (poloidal current) begins to rotate around the symmetry axis, each flux surface rotating with a particular angular frequency $\hat{\omega}_0$ ("isorotation"). Then this new state can be calculated by solving the corresponding ξ-equation in which $\hat{p}_0$ is increased by the energy density of the toroidal motion, $(1/2)\, \varrho_0 R^2 \hat{\omega}_0^2$ (R is the distance from the symmetry axis). This particular state which is characterized by the three flux functions $\hat{p}_0$, $\hat{I}_0$, and $\varrho_0 \hat{\omega}_0^2$ represents already the most general state since any poloidal motion can – after having obtained numerically the isorotation solution – be obtained by an appropriate gauge transformation $\xi(\Psi)$. In Sect. 5 the variational principle for the ξ-equation is derived, and Sect. 6 shows the following property of the gauge transformation: For any isorotation solution with $\hat{\omega}_0 \not\equiv 0$ there exists another solution with finite subcritical poloidal flow which minimizes the total energy of the system (keeping the flux surfaces fixed in space). This suggests the point of view that any isorotation solution is only a mathematical tool to define the flux surfaces; a real system will try to relax to a state with nonzero poloidal and toroidal velocity components.

The remaining part of the paper is devoted to the task of solving the ξ-equation numerically. The problem is, in the subcritical range, formally the same as in the static case: One has to solve a nonlinear elliptic boundary value problem. A particular difficulty is due to the possibility that several solutions may exist (bifurcation and parameter limits). This has been discussed several times (e.g. in [20]–[22]) and will not be treated here; a simple condition for the "source term" guarantees the uniqueness of a solution and is adopted here. Then the question remains how to estimate the numerical error introduced by iteration and discretization. In Sect. 7 we discuss a particular combination of both iteration and discretization: Since each iteration scheme assumes a good approximation (the preceding iteration, ξ^n), one knows already a good coordinate system defined by the flux contours of ξ^n and some radial lines crossing them [2]. Since also a variational principle is available one may discretize the problem by finite elements, improving the set of grid points at every step of the iteration. The matrix which has to be inverted in this step is then easily improved in order to solve the linearized problem exactly. This Newton iteration has been used in astrophysics long ago ("Henyey iteration" [23]), and also more recently [24] in the case of a parameter limit for the equilibrium. In Sect. 8 we solve a nonlinear analytical test problem numerically and compare the observed numerical error with theoretical predictions. It turns out that the iteration scheme using 14×28 grid points converts an initial mean error of 100 percent into 0.3 percent after 6 iterations.

2. Constraints for a Poloidal Incompressible Flow

Let us consider a stationary flow whose velocity components with respect to a cylindrical coordinate system (R, ζ, z) do not depend on the toroidal angle ζ. In the ideal case we have no sources and sinks of the fluid, and the continuity equation reads as follows:

$$\nabla \cdot (\varrho \boldsymbol{v}) = 0 . \tag{1}$$

This equation can be integrated by introducing two ζ-independent functions, say Ψ_0 and π_ζ, from which the momentum density can be deduced in the following way:

$$\boldsymbol{a} \equiv \varrho \boldsymbol{v} = (\nabla \zeta) \times \nabla \Psi_0 + \pi_\zeta \nabla \zeta . \tag{2}$$

Here, π_ζ is the z-component of the angular momentum density, and Ψ_0 can be interpreted by analogy with the magnetostatic case where the equation $\nabla \cdot \boldsymbol{B} = 0$ is integrated in a similar way: $\Psi_0(R, z)$ is $(1/2\pi)$ times the mass flux through a "rubber band" around the torus, defined by its boundaries, namely the circle $z =$ const, $R =$ const, and the inner symmetry line with distance R_m from the z-axis, $z = 0$, $R = R_\mathrm{m}$. From the solution of Ohm's law in the ideal case (with resistivity zero) it is well known that Ψ_0 and the corresponding magnetic potential Ψ (poloidal magnetic flux/2π) are functions of each other; therefore, the poloidal components $\boldsymbol{v}_\perp$ and $\boldsymbol{B}_\perp$ are proportional to each other, and $\boldsymbol{v}_\perp$ can be characterized completely by its Mach-number M with respect to the local Alfvén speed:

$$\boldsymbol{v}_\perp = M \frac{\boldsymbol{B}_\perp}{\sqrt{4\pi\varrho}} , \tag{3}$$

$$M = \frac{\mathrm{d}\Psi_0}{\mathrm{d}\Psi} \sqrt{4\pi/\varrho} . \tag{4}$$

Let us now evaluate the condition of incompressibility; due to axisymmetry it reads as follows:

$$\nabla \cdot \boldsymbol{v}_\perp = 0 . \tag{5}$$

From (1) and (2) we find then

$$0 = \varrho \boldsymbol{v}_\perp \cdot \nabla \varrho = \frac{\mathrm{d}\Psi_0}{\mathrm{d}\Psi} (\nabla\zeta) \cdot [(\nabla\Psi) \times \nabla\varrho] . \tag{6}$$

This means that the density ϱ is also a function of Ψ alone ("flux function"), and we will denote all flux functions from now on by a subscript zero. The same holds also for the poloidal Mach-number M, and our result reads then as follows:

$$\left.\begin{aligned} \varrho &= \varrho_0 \\ M &= M_0 \end{aligned}\right\} \text{for incompressible poloidal flow} . \tag{7}$$

If we assume that the stationary equilibrium is generated from a static equilibrium by "switching on" a velocity field according to (2), we may interpret $\varrho_0(\Psi)$ as the density profile of the static equilibrium. The only effect of the velocity field is then a deformation of the flux surfaces $\Psi =$ const, as we will see later. The incompressibility condition in hydrodynamics is usually well satisfied by flows with velocities far below the sound velocity. In the case of a tokamak plasma we find a similar result from a discussion of Bernoulli's law. In deriving the latter, one assumes often that the temperature is a flux function, i.e. $T = T_0$. Then the density cannot be a flux function in general; instead we find from (A.26) (neglecting the gravitational potential Φ, and putting $U_0 = T_0 \ln \varrho_0$):

$$T_0 \ln(\varrho/\varrho_0) + \boldsymbol{a}^2/2\varrho^2 + \omega_0 \pi_\zeta/\varrho = 0 . \tag{8}$$

Here, T_0 is the square of the thermal velocity, and ω_0 is a frequency (flux function) which constitutes one part of the toroidal velocity component. It should be noted here that (2) together with (A.6), (A.10), and (A.21) of the appendix allow the following representation of the momentum density $\boldsymbol{a}$:

$$\boldsymbol{a} = \frac{\mathrm{d}\Psi_0}{\mathrm{d}\Psi} \boldsymbol{B} - \varrho R^2 \omega_0 \nabla\zeta . \tag{9}$$

This relation holds for any equation of state; it implies that the stream lines are within the magnetic surfaces, and that the general velocity field is the superposition of a "parallel flow" ($\boldsymbol{v} \parallel \boldsymbol{B}$) and a rigid rotation of every flux surface around the z-axis, described by the two arbitrary flux functions Ψ_0 and ω_0. Now wee see that the assumption $\varrho = \varrho_0$ is, strictly speaking, not consistent with $T = T_0$; instead (8) allows the approximation $\varrho \approx \varrho_0$ if the following condition holds:

$$|\boldsymbol{a}^2/2\varrho_0^2 + \omega_0 \pi_\zeta/\varrho_0| \ll T_0 . \tag{10}$$

A sufficient condition for an incompressible poloidal flow is, therefore, $\boldsymbol{v}^2 \ll T_0$, i.e. the flow should be sufficiently subsonic with respect to the thermal velocity.

3. Equation for Flux Surfaces

It is convenient to express all flux functions as functions of an arbitrary function ξ, and to derive the scalar nonlinear equation for ξ from the equilibrium conditions. In this case the poloidal magnetic flux Ψ is also an arbitrary flux function $\Psi(\xi)$, in addition to the five others which I denote by $\Psi_0(\xi)$, $\omega_0(\xi)$, $\varrho_0(\xi)$, $p_0(\xi)$, and $I_0(\xi)$. The meaning of p_0 and I_0 has not yet been mentioned: p_0 is the pressure, and I_0 is R times the toroidal magnetic field of the associated static equilibrium, before "switching on" the gravitation and the velocities. The derivation of the equation for ξ has been given in the appendix for the usual assumption $T = T_0$. The result (A.36) reads as follows:

$$R^2(P' + \varrho\Phi') + Q' + (S/2)'(\nabla\xi)^2 + S\Delta_\perp^*\xi = 0\,. \quad (11)$$

The prime of any quantity, say F, means its total derivative with respect to ξ according to the following rule:

$$F' = \frac{\nabla\xi}{(\nabla\xi)^2}\nabla F\,. \quad (12)$$

Besides the gravitational potential Φ the following quantities appear in (11):

$$P = p + \boldsymbol{a}^2/2\varrho\,, \quad (A.29)$$

$$Q = \tfrac{1}{2}(I^2/4\pi - \pi_\zeta^2/\varrho)\,, \quad (A.33)$$

$$S = \Psi'^2/4\pi - \Psi_0'^2/\varrho\,. \quad (A.34)$$

The operator $\Delta_\perp^*$ has been defined in (A.23a, b). The difference with the static case can be described as follows: The pressure p has to be augmented by the kinetic energy of the flow, and the resulting quantity P is generally not a flux function; Q is R^2 times the difference of energies associated with the toroidal components of the magnetic field ($B_\zeta = I/R$) and the velocity, and Q is generally not a flux function either; S has no physical meaning since it contains $\Psi(\xi)$ which can be understood as an arbitrary transformation of the magnetic flux function Ψ. Here I want to mention the difference in the derivation of (11) for an incompressible flow. It is simply connected with the equation of state: Instead of (A.5) we have now

$$p = \varrho_0 T\,. \quad (13)$$

While (11) retains its form we have now a change of the form and of the meaning of Bernoulli's law. The only term which has now to be changed is the scalar product of $\boldsymbol{a}$ and the pressure gradient, (A. 25), in the following way:

$$\boldsymbol{a}\nabla p = \Psi_0'\{\xi, p\} = \Psi_0'\varrho_0\{\xi, p/\varrho_0\}\,, \quad (14)$$

where the bracket has been defined by (A.9). Therefore, we have simply to replace the term $T_0 \ln\varrho$ in Bernoulli's law (A.26) by (p/ϱ_0). Besides this we may define an "unperturbed pressure" p_0 associated with the static equilibrium (also with $\Phi = 0$) by $U_0 = p_0/\varrho_0$. Then the new Bernoulli law may be read as an equation of state for P, namely:

$$P = p_0 - \varrho_0\Phi - \omega_0\pi_\zeta\,. \quad (15)$$

It is now easy to evaluate all constraints also for the new case $\varrho = \varrho_0$; e.g. for π_ζ we find from (A.10), (A.18), and (A.21):

$$\pi_\zeta = \left(I_0\frac{\mathrm{d}\Psi_0}{\mathrm{d}\Psi} - R^2\varrho\,\omega_0\right)/(1 - M^2) \quad (16)$$

$$= \left(I_0\frac{\mathrm{d}\Psi_0}{\mathrm{d}\Psi} - R^2\varrho_0\,\omega_0\right)/(1 - M_0^2) \quad \text{for} \quad \varrho = \varrho_0\,.$$

Using this result in (15), we find for the first term in (11) the expression

$$R^2(P' + \varrho\Phi')$$

$$= R^2\left[p_0' - \varrho_0'\Phi - \left(\frac{\omega_0 I_0}{1 - M_0^2}\cdot\frac{\mathrm{d}\Psi_0}{\mathrm{d}\Psi}\right)' + \left(\frac{\varrho_0\,\omega_0^2 R^2}{1 - M_0^2}\right)'\right]. \quad (17)$$

It is also straight-forward to evaluate the second term in (11) for the case $\varrho = \varrho_0$; from (A.18) and (A.16) we find

$$Q' = \tfrac{1}{2}\left(\frac{I_0^2/4\pi}{1 - M_0^2}\right)' - \tfrac{1}{2}\left(\frac{\varrho_0\,\omega_0^2 R^4}{1 - M_0^2}\right)'. \quad (18)$$

An important observation for the practical solution of (11) is the following: While the prime acts also on the explicit R-dependence of P and Q (since $(\nabla R)\cdot\nabla\xi \neq 0$ in general) we have in fact a cancellation of these terms in the sum of (17) and (18), since

$$\tfrac{1}{2}(R^4)' = R^2(R^2)'\,. \quad (19)$$

Therefore, we obtain the following equation for the ξ-field in the case of a general incompressible flow:

$$S\Delta_\perp^*\xi + (S/2)'(\nabla\xi)^2 + \frac{R^2}{4\pi}\frac{\partial}{\partial\xi}V(R, \xi, \Phi) = 0 \quad (20)$$

with

$$V(R, \xi, \Phi) \tag{21}$$
$$= 4\pi \left[p_0 - \varrho_0 \Phi - \frac{\omega_0 I_0}{1 - M_0^2} \frac{\mathrm{d}\Psi_0}{\mathrm{d}\Psi} + \tfrac{1}{2} R^2 \varrho_0 \frac{\omega_0^2}{1 - M_0^2} \right] + \frac{1}{R^2} \frac{I_0^2/2}{1 - M_0^2} .$$

Note that the ξ-derivative of V acts now only on the flux functions p_0, ϱ_0, ω_0, I_0, and Ψ_0 (or M_0, respectively).

4. The Gauge for Subcritical Incompressible Flows

The freedom of choosing a function $\Psi(\xi)$ may be used to simplify the equilibrium condition (20). The usual equation for a static equilibrium corresponds to the choice $\Psi \equiv \xi$, which leads to the following expression for the quantity S, given by (A.34):

$$4\pi S \equiv 1 . \tag{22}$$

In the case of an incompressible flow S becomes a flux function, and if the flow is subcritical in the sense of $M_0^2 < 1$ we may still maintain (22) as a gauge condition for $\Psi(\xi)$ or $\xi(\Psi)$ (otherwise the sign of S is changed). From (A.34) we find then the following relation (up to an arbitrary constant):

$$\xi = \int \mathrm{d}\Psi \sqrt{1 - M_0^2} . \tag{23}$$

For this particular choice of the ξ-field we obtain then the equilibrium condition

$$\Delta_\perp^* \xi + R^2 \frac{\partial}{\partial \xi} V(R, \xi, \Phi) = 0 \tag{24}$$

with

$$V(R, \xi, \Phi) = 4\pi [\hat{p}_0 - \varrho_0 \Phi + \tfrac{1}{2} R^2 \varrho_0 \hat{\omega}_0^2] + (1/R^2)\, \hat{I}_0^2/2 , \tag{25}$$

$$\hat{p}_0 = p_0 - \sqrt{\varrho_0/4\pi}\, \hat{\omega}_0 \hat{I}_0 M_0 , \tag{26}$$

$$\hat{I}_0 = I_0/\sqrt{1 - M_0^2} , \tag{27}$$

$$\hat{\omega}_0 = \omega_0/\sqrt{1 - M_0^2} . \tag{28}$$

Let us discuss the result in the absence of gravitation. A solution of (24) is then already specified by giving three flux functions: $\hat{p}_0$, $\hat{I}_0$, and $\varrho_0 \hat{\omega}_0^2$. In the case of $\omega_0 = 0$ we recover exactly the static problem for ξ (instead of Ψ), depending on the functions p_0 and $\hat{I}_0$ (instead of p_0 and I_0). So we may run a static code and re-interpret the result for an arbitrary choice of the flux function M_0, with $M_0^2 < 1$. On the other hand, for $M_0 = 0$ we find the Ψ-equation for the case of purely toroidal rotation where every flux surface rotates as a rigid body with frequency ω_0*: p_0 has to be enhanced by the kinetic energy density of the toroidal motion, $\frac{1}{2}\varrho_0 R^2 \omega_0^2$. Surprisingly – because the problem is non-linear – this represents already the general case of a subcritical incompressible flow: The poloidal Mach-number can be completely transformed away by putting $p_0 \to \hat{p}_0$, $I_0 \to \hat{I}_0$, $\omega_0 \to \hat{\omega}_0$, and $\Psi \to \xi$. Having specified the three functions $\hat{p}_0$, $\hat{I}_0$, and $\varrho_0 \hat{\omega}_0^2$, we can, in principle, solve the ξ-equation. One particular solution then specifies twice an infinity of physical solutions: We may afterwards specify ϱ_0 and M_0 and calculate the associated functions p_0, I_0, and ω_0. For a positive $\hat{p}_0$ we will always find a positive p_0 if $\hat{\omega}_0 \hat{I}_0 M_0 \geqq 0$. On physical reasons ($\Phi \leqq 0$ for a self-gravitating system) we will then always have the case $V \geqq 0$.

5. Variational Principle and Uniqueness of Solutions

For the numerical solution of partial differential equations it is often helpful to find a variational formulation. Consider the following functional of ξ:

$$W[\xi] = \frac{1}{2\pi} \int_V \mathrm{d}^3x \left[\frac{1}{2R^2} (\nabla\xi)^2 - V(R, \xi, \Phi) \right] . \tag{29}$$

The volume V is assumed to be generated by the rotation of a poloidal area F around the axis of symmetry (F should not contain this axis). Since all functions are considered to be independent of ζ we may also use the replacement

$$\frac{1}{2\pi} \int_V \mathrm{d}^3x \to \iint \mathrm{d}z\, \mathrm{d}R R . \tag{30}$$

The first variation of W is given as follows:

$$\delta W = \frac{1}{2\pi} \int_V \mathrm{d}^3x \left[\frac{1}{R^2} (\nabla\xi)\, \delta\nabla\xi - \frac{\partial V}{\partial \xi} \delta\xi \right] . \tag{31}$$

Permuting δ and ∇ in the first term, and performing a partial integration with vanishing contribution at the boundary ∂V (since there $\delta\xi$ should vanish) we find, using (A.23 a):

$$\delta W = -\frac{1}{2\pi} \int_V \mathrm{d}^3x \left[\frac{1}{R^2} \Delta_\perp^* \xi + \frac{\partial V}{\partial \xi} \right] \delta\xi . \tag{32}$$

* Strictly speaking, the frequency is $-\omega_0$, see (9).

The vanishing of the right-hand side for arbitrary $\delta\xi$ implies the equilibrium condition, (24), for ξ. The second variation is

$$\delta^2 W = \frac{1}{2\pi} \int_V d^3x \left[\frac{1}{R^2} (\delta \nabla \xi)^2 - \frac{\partial^2 V}{\partial \xi^2} (\delta \xi)^2 \right]. \quad (33)$$

Obviously we have $\delta^2 W > 0$, i.e. a minimum of W for the solution of (24), if the following condition holds:

$$\frac{\partial^2}{\partial \xi^2} V(R, \xi, \Phi) \leqq 0 . \quad (34)$$

It should be noted that the same condition is also a sufficient condition for the uniqueness of the solution ξ (see, e.g. p. 322/323 of [25]). Let us also note that $W[\xi]$ is not the energy functional whose variation with respect to a displacement vector $\boldsymbol{\xi}$ is known to be stationary for a static equilibrium [1]; instead, it is the difference between several parts of the energy of the corresponding isorotation solution ($M_0 = 0$), and the scalar field ξ can be varied without constraints (besides $\delta\xi|_{\partial V} = 0$).

6. The Energy Functional

In this section we consider the energy of equilibria with the same nested flux surfaces $\xi = \text{const}$, and with the same flux functions $\hat{p}_0$, $\hat{I}_0$, $\hat{\omega}_0$, and ϱ_0; only the poloidal Alfvén Mach number M_0 varies. Let us first compare the energy associated with the poloidal components of $\boldsymbol{v}$ and $\boldsymbol{B}$:

$$E_\perp = \int d^3x \left(\frac{\varrho_0}{2} \boldsymbol{v}_\perp^2 + \boldsymbol{B}_\perp^2 / 8\pi \right)$$
$$= \int d^3x \frac{1}{8\pi R^2} \frac{1 + M_0^2}{1 - M_0^2} (\nabla \xi)^2 . \quad (35)$$

In the last step we used (A.6), (A.7), (4), (7), and (23). In the subcritical range $0 \leqq M_0^2 \leqq 1$ we obtain obviously a monotonically increasing function of M_0^2. In order to derive an expression for the energy $E_\parallel$ of the corresponding toroidal components, the following equation for $I \equiv R B_\zeta$ is useful:

$$I = (\hat{I}_0 - M_0 R^2 \hat{\omega}_0 \sqrt{4\pi\varrho_0}) / \sqrt{1 - M_0^2} . \quad (36)$$

Here we have used (A.18), (16), (4), (7), (27), and (28). Then (A.6), (A.7), (16), and (36) lead to the relations

$$E_\parallel = \int d^3x \left(\frac{\varrho_0}{2} v_\zeta^2 + B_\zeta^2 / 8\pi \right)$$
$$= \int d^3x \frac{1}{8\pi R^2} [(4\pi/\varrho_0)\, \pi_\zeta^2 + I^2]$$
$$= \int d^3x \frac{1}{8\pi R^2 (1 - M_0^2)}$$
$$\cdot \{(1 + M_0^2)(\hat{I}_0^2 + R^4 4\pi\varrho_0 \hat{\omega}_0^2) - 4 M_0 R^2 \sqrt{4\pi\varrho_0}\, \hat{I}_0 \hat{\omega}_0\} . \quad (37)$$

In the last equation we find two contributions in the curly bracket: The first one has the same M_0-dependence as $E_\perp$ while the second one is monotonically decreasing with increasing $|M_0|$ ($\leqq 1$) if the following condition holds:

$$M_0 \hat{I}_0 \hat{\omega}_0 > 0 . \quad (38)$$

This offers the possibility of minimizing the total energy on any flux surface individually by varying M_0. Since the internal energy and the gravitational energy are independent of M_0, we have only to add (35) and (37) and to write the result as an integral with respect to ξ, namely:

$$E_\perp + E_\parallel = \int d\xi\, E_0 , \quad (39)$$

$$E_0 = \frac{1 + M_0^2}{1 - M_0^2} F_0 - \frac{2 M_0}{1 - M_0^2} G_0 , \quad (40)$$

$$F_0 = 2\pi \oint_{\xi = \text{const}} \frac{R\, dl}{|\nabla \xi|} \cdot \frac{1}{8\pi R^2} \{(\nabla\xi)^2 + \hat{I}_0^2 + R^4 4\pi\varrho_0 \hat{\omega}_0^2\} , \quad (41)$$

$$G_0 = 2\pi \oint_{\xi = \text{const}} \frac{R\, dl}{|\nabla \xi|} \sqrt{\frac{\varrho_0}{4\pi}}\, \hat{I}_0 \hat{\omega}_0 . \quad (42)$$

Note that F_0 and G_0 are known flux functions which are kept fixed; dl is the line element of a particular closed poloidal curve $\xi = \text{const.}$; F_0 is the energy per interval $d\xi$ of the associated isorotation solution. From (41) and (42) we have

$$F_0 - |G_0| > 2\pi \oint_{\xi = \text{const}} \frac{R\, dl}{|\nabla \xi|} \frac{1}{8\pi R^2} \cdot \{\hat{I}_0^2 - 2 R^2 \sqrt{4\pi\varrho_0}\, |\hat{I}_0 \hat{\omega}_0| + R^4 4\pi\varrho_0 \hat{\omega}_0^2\} \geqq 0 . \quad (43)$$

It is then easily verified that the following expression is a subcritical poloidal Alfvén Mach number which leads to a minimum of E_0:

$$M_0 = (F_0/G_0)[1 - \sqrt{1 - (G_0/F_0)^2}] . \quad (44)$$

A straight-forward evaluation of (40) leads then to the following minimum of E_0:

$$(E_0)_{\min} = \sqrt{1 - (G_0/F_0)^2}\, F_0 \,. \tag{45}$$

Thus the energy of the isorotation solution can considerably be lowered if $|G_0|$ is not much smaller than F_0; this may happen according to the inequality (43) if

$$|\hat{I}_0| \approx R_0^2 \sqrt{4\pi\varrho_0}\, |\hat{\omega}_0| \,, \tag{46}$$

where R_0 is an average distance of the flux surface from the z-axis. This, however, means that the purely toroidal motion reaches the order of the toroidal Alfvén velocity, a case which is only of interest for a high-beta plasma. In the case of a tokamak we have to expect $|G_0|/F_0 \ll 1$, and, therefore, the effect of the poloidal velocity will be small.

7. Numerical Methods for Solving the ξ-Equation

Two essential steps are necessary before a numerical solution of (24) can be obtained: Since the equation is nonlinear one has to find a sufficiently stable iteration scheme; secondly, one has, of course, to use a finite representation of $\boldsymbol{x}$-space. A further problem is posed by bifurcation; but here we assume that the inequality (34) holds, then we are sure that the solution is unique. Let us first write (24) in a short-hand manner without noting the explicit R- and Φ-dependence:

$$H(\xi) \equiv \Delta_\perp^* \xi + f(\xi) = 0 \tag{47}$$

with

$$f(\xi) = R^2 \frac{\partial}{\partial \xi} V(R, \xi, \Phi) \,, \tag{48}$$

$$\partial f/\partial \xi \leqq 0 \,. \tag{49}$$

Assuming that the n^{th} step of an iteration leads to ξ^n as an approximation for the true solution ξ^* we define an ideal increment $\delta\xi$ as follows:

$$0 = H(\xi^n + \delta\xi) \equiv H(\xi^n) + \Delta_\perp^* \delta\xi + f(\xi^n + \delta\xi) - f(\xi^n) \,. \tag{50}$$

In practice, however, (50) is solved after linearization with respect to $\delta\xi$; this defines the equation for the iteration step which leads to ξ^{n+1} ($\approx \xi^n + \delta\xi$), namely:

$$L_n \xi^{n+1} + F(\xi^n) = 0 \tag{51}$$

with

$$L_n = \Delta_\perp^* + \frac{\partial f(\xi_n)}{\partial \xi_n} \,, \tag{52}$$

$$F(\xi^n) = f(\xi_n) - \xi_n \frac{\partial f(\xi_n)}{\partial \xi_n} \,. \tag{53}$$

Let us see how the error is iterated by this procedure. We define [20]

$$\varepsilon_n = \xi^n - \xi^* \tag{54}$$

as the error of the n^{th} step; then we find after some algebra to lowest order in ε_n:

$$\varepsilon_{n+1} = (1/2)\, L_n^{-1} \frac{\partial^2 f(\xi^*)}{\partial \xi^{*2}}\, \varepsilon_n^2 \,. \tag{55}$$

The iteration error decreases quadratically in ε_n, as was expected; a further evaluation of ε_{n+1} is given in the next section.

To get a finite representation of the iteration scheme we first formulate (51) as a variational problem:

Find ξ^{n+1} with $I^n(\xi^{n+1}) \leqq I^n(\xi)$
for every choice of ξ and with

$$I^n(\xi) = \int \frac{1}{R} \left[\frac{(\nabla\xi)^2}{2} - \frac{1}{2} \frac{\partial f}{\partial \xi^n}(\xi^n)\, \xi^2 - F(\xi^n)\, \xi \right] \mathrm{d}R\, \mathrm{d}z \,. \tag{56}$$

One immediately sees that the functions ξ admissible to this formulation only need to have a L^2-integrable[1] first derivative (and must no longer be twice differentiable), or more mathematically spoken: We can enlarge the space of functions admissible to this problem to the Sobolev-space[2] H_0^1; minimizing I^n in the enlarged space H_0^1 leads then to the so-called weak solutions.

For the numerical approach we restrict the space of admissible functions H_0^1 to a finite dimensional subspace V_h characterized by its basis functions $h_i(R, z)$.

So every $\xi_h \in V_h$ has a representation

$$\xi_h = \sum_i \xi_i h_i(R, z) \,. \tag{57}$$

---

[1] L^2-integrable means $\sqrt{\int_V R\, \mathrm{d}R\, \mathrm{d}z\, (\cdot)^2} =: \|\cdot\|_{L^2} < \infty$.

[2] $f \in H^1$ if
$\sqrt{\int_V R\, \mathrm{d}R\, \mathrm{d}z\, [|f(R,z)|^2 + |f_R(R,z)|^2 + |f_z(R,z)|^2]} < \infty$;
$f \in \mathrm{H}_0^1$ if $f \in \mathrm{H}^1$ and $f|_{\partial V} = 0$.

Inserting this into (56) we get

$$I_h^n(\xi_h) = \sum_{i,j} M_{ij}^n \xi_i \xi_j + \sum_i K_i^n \xi_i \tag{58}$$

with

$$M_{ij}^n = \int \left[\frac{\nabla h_i \cdot \nabla h_j}{2R} - \frac{1}{2R} \frac{\partial f}{\partial \xi} (\xi_h^n) \, h_i h_j \right] \mathrm{d}R \, \mathrm{d}z \, ,$$

$$K_i^n = - \int h_i F(\xi_h^n) \frac{\mathrm{d}R}{R} \mathrm{d}z \, , \tag{59}$$

where M_{ij}^n, K_i^n depend on the preceding iterate ξ_h^n in V_h.

For (58) the minimization procedure leads to

$$\sum_j 2 M_{ij}^n \xi_j^{n+1} + K_i^n = 0 \quad \forall_i \, , \tag{60}$$

as the equation for the coefficients ξ_j^{n+1} of the $(n+1)^{\text{th}}$ iterate. Usually this method of discretizing a minimization problem is called Ritz-method. The index h of the finite dimensional subspace represents the particular choice of the subspace V_h which we choose as a finite element space. We construct V_h by first discretizing the integration domain into quadrangles as shown in Figure 1. The edges of each quadrangle are given by the grid points (R_i, z_i). Due to the finite element method we use basis functions $h_i(R, z)$ vanishing identically outside the quadrangles surrounding (R_i, z_i), and reaching unity at the referred gridpoint. So the coefficients in (57) are the nodal values of ξ at the gridpoints.

We choose quadrangular elements to discretize the domain because they are well suited to represent the nested contour lines of ξ. In general, each quadrangle looks different; consequently we have different basis functions with different supports.

Because we want to perform the integration of the M_{ij} and K_i in an efficient way we use the method of isoparametric bilinear finite elements [26], i.e. the integrations are performed separately on each quadrangle where the global (R, z)-coordinates for each particular quadrangle are obtained by a bilinear transformation acting in local (s, t)-coordinates on a unit square (Fig. 2). If $P_k = (R_k, z_k)$, $k = 1, \ldots, 4$, are the four edges of the considered quadrangle we set

$$R(s,t) = \sum_{k=1}^{4} R_k N_k(s,t) \, ; \quad N_1(s,t) = (1-s)(1-t) \, ,$$
$$N_2(s,t) = s(1-t) \, , \tag{61 a}$$
$$z(s,t) = \sum_{k=1}^{4} Z_k N_k(s,t) \, ; \quad N_3(s,t) = st \, ,$$
$$N_4(s,t) = (1-s)\, t \, .$$

The (s, t)-representation of $\xi_h \in V_h$ restricted to the considered quadrangle is prescribed in the same way

$$\xi(s,t) = \sum_{k=1}^{4} \xi_k N_k(s,t) \, . \tag{61 b}$$

That means that $h_i(R, z)$ restricted to the domain of a quadrangle neighbouring (R_i, z_i) has one of the $N_k(s, t)$ as local coordinate representation. Due to this composition of the h_i by the N_k we notice the necessity of enlarging the admissible function space to H_0^1 because every h_i is within H_0^1, but none is twice differentiable in the classical sense (which would be needed for a similar restriction of the admissible function space on (51)).

Now we can transform the integrals (59) to local (s, t) coordinates which is of great advantage for programming the code independent of a special discretization. The transformation is obtained with the help of

$$\mathrm{d}R \, \mathrm{d}z = J \, \mathrm{d}s \, \mathrm{d}t \, ; \quad J = \det \begin{pmatrix} \frac{\partial R}{\partial s} & \frac{\partial R}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{pmatrix} ,$$

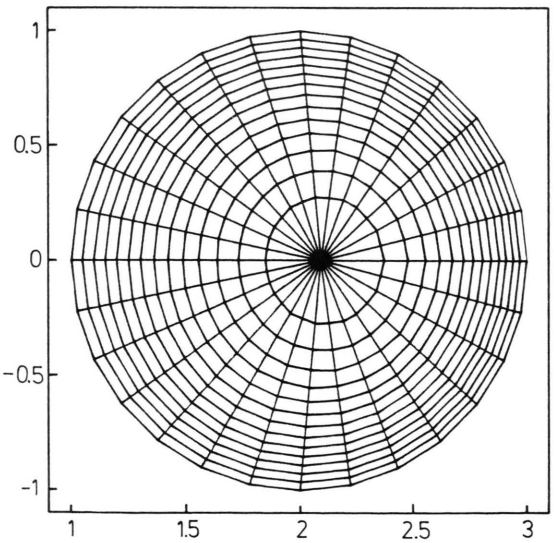

Fig. 1. Discretization of the domain by rearranged finite elements in the (R, z)-plane.

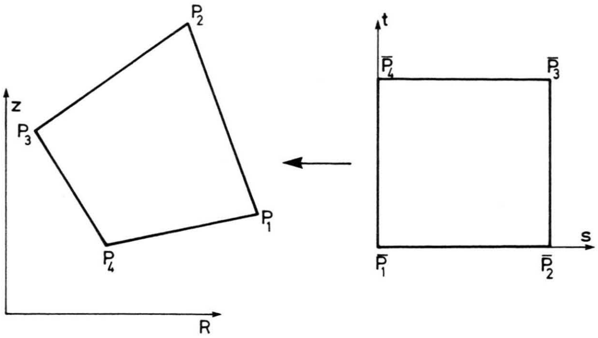

Fig. 2. Isoparametric transformation from local (s, t)-coordinates to the global (R, z)-coordinates.

and

$$\begin{pmatrix} \dfrac{\partial}{\partial R} \\ \\ \dfrac{\partial}{\partial z} \end{pmatrix} = \frac{1}{J} \begin{pmatrix} \dfrac{\partial z}{\partial t} & -\dfrac{\partial z}{\partial s} \\ \\ -\dfrac{\partial R}{\partial t} & \dfrac{\partial R}{\partial s} \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial s} \\ \\ \dfrac{\partial}{\partial t} \end{pmatrix}.$$

To exhaust the approximation properties of the finite elements we rearrange the grid after each iteration step, so that every shell of the grid corresponds to a contour line of the approximate solution. This is useful for solutions having strong gradients.

8. Convergence Properties and Test Problem

To investigate the convergence of our discrete iteration scheme (60), one has to take a closer look at the distance between the solution of the whole problem ξ^* and the n^{th} iterate of the discrete scheme (60) ξ_{d}^n in a suitable norm; we choose the L_2-norm which is possible since our admissible function space is H_0^1.

We put:

$$\bar{\varepsilon}^n = \| \xi_{\text{d}}^n - \xi^* \|_{L_2} .$$

If ξ_{d}^* is the solution of (60) we can estimate $\bar{\varepsilon}^n$:

$$\begin{aligned} \bar{\varepsilon}^n &= \| \xi_{\text{d}}^n - \xi_{\text{d}}^* + \xi_{\text{d}}^* - \xi^* \|_{L_2} \\ &\leqq \| \xi_{\text{d}}^n - \xi_{\text{d}}^* \|_{L_2} + \| \xi_{\text{d}}^* - \xi^* \|_{L_2} \end{aligned}$$

by the two errors $\varepsilon^* = \| \xi_{\text{d}}^* - \xi^* \|_{L_2}$ and $\varepsilon_{\text{d}}^n = \| \xi_{\text{d}}^n - \xi_{\text{d}}^* \|_{L_2}$, which we investigate separately.

The distance ε^* of the true solution to the approximate one is obtained by the discretization of (51). This error should vanish if the number N of elements goes to infinity. Rappaz [27] has investigated ε^* for similar problems deduced from some abstract mathematical theorems. He got

$$\varepsilon^* < c \cdot h^2 , \tag{62}$$

where h is a typical mesh length of the discretization. With N defined as above one sees $h \sim N^{-1/2}$. So we obtain

$$\varepsilon^* < K_1/N \tag{63}$$

as an estimate. We tried to find such an estimate for ε^* with our code experimentally by integrating the analytically known test problem

$$\Delta_\perp^* \xi - 4 \xi \ln \xi - 2 \left(1 + \frac{2}{R} \right) \xi = 0 \tag{64}$$

with the solution

$$\xi = \exp [(R-2)^2 + z^2] .$$

By taking a sufficiently large number of iterations we found from linear regression the following empirical formula (see Fig. 3):

$$\varepsilon^* = \frac{a}{N} + b \quad \text{with} \quad a \approx 1.6 , \quad b = 0.00012 . \tag{65}$$

The point (0, 0) in Fig. 3 has not been included in the linear regression, but b turns out to be of the same order as the mean deviation of the measured points from the straight line; therefore, it can be neglected, and the inequality (62) is verified.

One peculiarity of our numerical code was that we found the error at the central grid point bigger than on all other points by a factor 10. This effect roots in the rearrangement of the grid to the lines $\xi = \text{const}$. Namely, the element size connected with this procedure is then $\Delta l = \Delta \xi / |\nabla \xi|$; with N going to infinity, the decrease of the element size at the center of the grid near the magnetic axis (where $\nabla \xi = 0$) is an order of magnitude slower than at the other elements. Naturally this effects the big error at the central grid point. To diminish this deficiency we improved the code by setting the central ξ-value equal to that of the innermost contour line, and thus additionally warranted $\nabla \xi = 0$ at the center of the grid (i.e. the estimate for the locus of the magnetic axis). The gain of accuracy for our test problem is

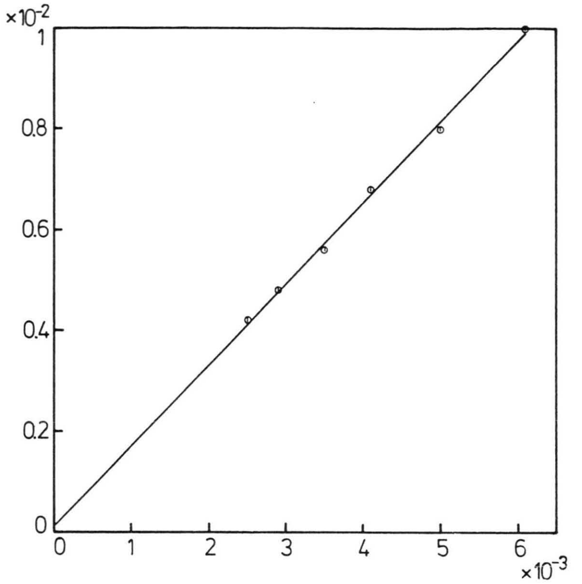

Fig. 3. Discretization error ε^* as a function of 1/N; N = Number of elements.

enormous and can be seen by comparing runs I, III, of Table 1.

To investigate the other part of the total error ε_d^n we notice that ξ_d^*, $\xi_d^n \in V_h$; therefore, the L_2-norm defining ε_d^n can be estimated by the corresponding Euklidean distance $|\boldsymbol{\varepsilon}_d^n|_E$ in the space of the nodal values R^{N+1} (see[3]); this allows the transition to the discrete scheme in the $\bar{\varepsilon}^n$-estimate:

$$(\varepsilon_d^n)^2 = \int |\xi_d^* - \xi_d^n|^2 R \, dR \, dz$$

$$= \int \left| \sum_{i=1}^{N+1} (\xi_i^* - \xi_i^n) h_i \right|^2 R \, dR \, dz$$

$$= \sum_{i,j=1}^{N+1} (\xi_i^* - \xi_i^n)(\xi_j^* - \xi_j^n) \int h_i h_j R \, dR \, dz$$

$$= \sum_{i,j=1}^{N+1} \varepsilon_{di}^n \varepsilon_{dj}^n N_{ij} \leqq C \sum_{i=1}^{N+1} |\varepsilon_{di}^n|^2 = C \, |\boldsymbol{\varepsilon}_d^n|_E^2 . \quad (66)$$

For the solution of the discrete iteration we can now make a similar analysis as we did in Sect. 7 for ε^n. After some algebra we obtain up to an accuracy of order two

$$|\boldsymbol{\varepsilon}_d^{n+1}|_E \leqq K \, |\boldsymbol{\varepsilon}_d^n|_E^2 \quad (67)$$

[3] N elements represent $N+1$ grid points.

Table 1. Evaluation of the error $\bar{\varepsilon}^n$ for runs with different specifications. During run I and II, the central value is set equal to that of the innermost contour line after every iteration, whereas there is no special handling of the central value in run III and IV. Additionally run I and III are obtained with grid-rearrangement after every iteration. The other two runs are made with fixed grids obtained from the related runs with rearrangement. The error $\bar{\varepsilon}^0$ is the distance of the initial guess ξ = const. = e to the true solution.

Number of Iterations	Run I	Run II	Run III	Run IV
0	1.96029	1.9597	1.96029	1.9597
1	0.05158	0.03599	0.050827	0.033102
2	0.01888	0.003874	0.021758	0.008953
3	0.008275	0.004346	0.010118	0.008354
4	0.004612	0.004276	0.0087368	0.008357
5	0.004359	–	0.0083764	–
6	0.004280	–	0.0083575	–
7	0.004281	–	0.0083575	–

with K depending on V_h and f. Before we can start verifying (67) for our code we have to take into account that the central grid point correction as well as the grid rearrangement are not included in the analysis leading to (67). So we made a test run without rearrangement and central grid point correction (see Table 1, run IV). Beginning with a sufficiently small $\bar{\varepsilon}^n$ (for run IV: $\bar{\varepsilon}^1$), we found K to be of order 1 for run IV using $\varepsilon_d^n = \bar{\varepsilon}^n - \varepsilon^*$. This is the same order that we obtained by estimating the factor of ε_n^2 in (55). So we are sure that the code works well though the effects of grid rearrangement and special handling of the central ξ-value are not included in the formula for the total error $\bar{\varepsilon}^n$.

9. Summary

Stationary axisymmetric plasma configurations can be calculated with ordinary codes for static equilibria if the velocity field is sufficiently smaller than the local sound velocity. The nested flux surfaces can generally be obtained from a particular solution with purely toroidal motion ("isorotation"); the general solution with the same flux surfaces is then obtained by a "gauge transformation" corresponding to nonzero poloidal velocity components. In particular, we find analytically the poloidal Alfvén Mach number (Eq. (44)) which minimizes the total energy; the minimum may be strongly pronounced in a high-beta plasma. This result could also be of interest in solar physics: The photosphere shows – besides a toroidal motion – also meridional circulations. Comparing (16), (36), and (38), we find the following sign rule: If the motion of the isorotation solution is antiparallel to the toroidal magnetic field ($\hat{I}_0 \hat{\omega}_0 > 0$), then the minimizing poloidal velocity is parallel to $\boldsymbol{B}_\perp$ ($M_0 > 0$) and vice versa.

The codes for calculating flux surfaces are most effective if the grid points ly on these surfaces [2]; then the matrix M which has to be inverted at every iteration step is also iteratively improved, and this can be done by removing the linear iteration error exactly to zero. A variational principle allows to discretize the problem by finite elements. Scaling properties of both the iteration and discretization error are obtained and compared with the results for a nonlinear analytical test problem. It turns out that the total error is further diminished if the ξ-value of the solution at the central grid point is approximated by its value at the innermost contour line; this is plausible since the discretization error is relatively large near an extremum of the solution.

To apply this technique of calculating equilibria one needs empirical or theoretical information on the three flux functions $\hat{p}_0$, $\hat{I}_0$, and $\varrho_0 \hat{\omega}_0^2$. Astrophysical objects with self-gravitation can also be treated for a given density distribution ϱ_0 by solving simultaneously the Poisson equation for the gravitational potential Φ.

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Appendix A

Here we derive the complete set of equations characterizing an axisymmetric stationary plasma equilibrium. Using the abbreviations

$$\boldsymbol{a} = \varrho\, \boldsymbol{v}\,; \quad \boldsymbol{J} = \nabla \times \boldsymbol{B}\,,$$

we have to fulfill the following set of stationary ideal MHD-equations:

$$\nabla \cdot (\boldsymbol{v}\, \boldsymbol{a}) + \nabla p = \frac{1}{4\pi} \boldsymbol{J} \times \boldsymbol{B} - \varrho \nabla \Phi\,, \tag{A.1}$$

$$c\, \nabla \Phi_0 = \boldsymbol{v} \times \boldsymbol{B}\,, \tag{A.2}$$

$$\nabla \cdot \boldsymbol{B} = 0\,, \tag{A.3 a}$$

$$\nabla \cdot \boldsymbol{a} = 0\,. \tag{A.3 b}$$

Equation (A.1) is the momentum balance between the inertia term, the pressure gradient, the Lorentz and the gravitational force; (A.2) is Ohm's law for the electric potential Φ_0, and (A.3) state that the vector fields $\boldsymbol{B}$ and $\boldsymbol{a}$ are free of sources.

Instead of prescribing an energy law we assume that it is consistent with a temperature T_0 which is constant on magnetic surfaces:

$$\boldsymbol{B}\cdot\nabla T_0=0\,. \tag{A.4}$$

In fact, T_0 denotes the square of a thermal velocity since we will also use the ideal gas law in the notation

$$p=\varrho T_0\,. \tag{A.5}$$

In the cylindrical coordinate system (R,ζ,z) Eqs. (A.3) are solved by introducing four ζ-independent functions Ψ, I, Ψ_0, and π_ζ, from which the vector fields $\boldsymbol{B}$, $\boldsymbol{a}$, can be constructed in the following way:

$$\boldsymbol{B}=(\nabla\zeta)\times\nabla\Psi+I\nabla\zeta\,, \tag{A.6}$$

$$\boldsymbol{a}=(\nabla\zeta)\times\nabla\Psi_0+\pi_\zeta\nabla\zeta\,. \tag{A.7}$$

a) Solution of Ohm's Law

Introducing (A.6, 7) into (A.2), we find

$$\varrho c\nabla\Phi_0=(I/R^2)\nabla\Psi_0-(\pi_\zeta/R^2)\nabla\Psi+\{\Psi_0,\Psi\}\nabla\zeta\,, \tag{A.8}$$

with

$$\{\Psi_0,\Psi\}=[(\nabla\Psi_0)\times(\nabla\Psi)](\nabla\zeta)\,. \tag{A.9}$$

Due to the axisymmetry of the problem $(\partial/\partial\zeta=0)$ the last term in (A.8) has to vanish separately; therefore, the potentials Ψ_0 and Ψ are functions of each other. In order to save the symmetry between magnetic and fluid quantities we assume that both potentials Ψ_0 and Ψ are functions of a scalar function ξ. Then (A.8) states that also Φ_0 depends on ξ only, and we obtain the relation

$$\varrho c\Phi_0'=(I\Psi_0'-\pi_\zeta\Psi')/R^2\,, \tag{A.10}$$

where the prime denotes the differentiation with respect to ξ. Two cases can be distinguished:

α) $\boldsymbol{\Phi_0'\neq 0}$

This is the ordinary case which will be considered in more detail. Then (A.10) is used as an equation to determine the density ϱ.

β) $\boldsymbol{\Phi_0'=0}$

In this particular case we find from (A.10) an equation for π_ζ:

$$\pi_\zeta=I\frac{d\Psi_0}{d\Psi}\,. \tag{A.11}$$

This means according to (A.6, 7) that the velocity vector $\boldsymbol{v}$ is everywhere parallel to $\boldsymbol{B}$; in particular, we find

$$\boldsymbol{v}=M\frac{\boldsymbol{B}}{\sqrt{4\pi\varrho}} \tag{A.12}$$

with the following expression for the Alfvén Mach-number:

$$M=\frac{d\Psi_0}{d\Psi}\sqrt{4\pi/\varrho}\,. \tag{A.13}$$

b) *Toroidal momentum balance*

Using the identity

$$\times(\nabla\times\boldsymbol{v})=\varrho\nabla v^2/2-(\boldsymbol{a}\cdot\nabla)\,\boldsymbol{v}\,,$$

we obtain the following form of the momentum balance (A.1):

$$\nabla(p+(\varrho/2)\,v^2)+\left(\boldsymbol{a}\cdot\nabla\frac{1}{\varrho}\right)\boldsymbol{a}+\boldsymbol{\omega}\times\boldsymbol{a}/\varrho=(a^2/2)\,\nabla\frac{1}{\varrho}+\frac{1}{4\pi}\boldsymbol{J}\times\boldsymbol{B} \tag{A.14}$$

with

$$\boldsymbol{\omega}=\nabla\times\boldsymbol{a}\,. \tag{A.15}$$

After scalar multiplication of (A.14) with $\nabla\zeta$ the following three expressions survive (using (A.6–9)):

$$\left(\boldsymbol{a}\cdot\nabla\frac{1}{\varrho}\right)(\boldsymbol{a}\cdot\nabla\zeta)=\Psi_0'(\pi_\zeta/R^2)\left\{\xi,\frac{1}{\varrho}\right\},$$

$$(\boldsymbol{\omega}\times\boldsymbol{a}/\varrho)\cdot\nabla\zeta=(\Psi_0'/\varrho R^2)\{\xi,\pi_\zeta\}\,,$$

$$\frac{1}{4\pi}(\boldsymbol{J}\times\boldsymbol{B})\cdot\nabla\zeta=(\Psi'/4\pi R^2)\{\xi,I\}\,.$$

The curly brackets at the right-hand side are defined in (A.9). Since Ψ_0 and Ψ are functions of ξ only ("flux functions"), we obtain then the following momentum balance in the toroidal direction:

$$\frac{1}{R^2}\left\{\xi,\left(\frac{\Psi_0'\pi_\zeta}{\varrho}-\frac{\Psi'}{4\pi}I\right)\right\}=0\,. \tag{A.16}$$

Therefore, we find an additional flux function which we denote by $K_0\equiv K_0(\xi)$:

$$\Psi_0'\pi_\zeta/\varrho-\Psi' I/4\pi=K_0\,. \tag{A.17}$$

In the magnetic case $(\Psi'\neq 0)$ this equation serves to eliminate the poloidal current I; we put

$$K_0=-(\Psi'/4\pi)\,I_0\,,$$

and obtain the relation

$$I = I_0 + (4\pi/\varrho)\frac{d\Psi_0}{d\Psi}\pi_\zeta . \tag{A.18}$$

c) *Bernoulli's law*

Let the subscript $\perp$ denote the poloidal component of any vector (i.e. its projection into the R, z-plane). From the solution of Ohm's law and (A.6, 7) we know that $\boldsymbol{a}_\perp$ and $\boldsymbol{B}_\perp$ are proportional to each other, namely:

$$\boldsymbol{a}_\perp = \frac{d\Psi_0}{d\Psi}\boldsymbol{B}_\perp . \tag{A.19}$$

In addition, both vectors are perpendicular to the gradient of ξ and of any flux function. From (A.6, 7, 10, 19) we obtain the most general form for $\boldsymbol{a}$ in the "magnetic case" ($\Psi' \neq 0$), namely:

$$\boldsymbol{a} = \frac{d\Psi_0}{d\Psi}\boldsymbol{B} - R^2\varrho\omega_0\nabla\zeta , \tag{A.20}$$

with

$$\omega_0 = c\frac{d\Phi_0}{d\Psi} . \tag{A.21}$$

The most general symmetric stationary velocity field is, therefore, specified by $\boldsymbol{B}$, ϱ, and the two flux functions Ψ_0 and ω_0. The last arbitrary flux function can now be found from the component of the momentum balance equation (A.14) parallel to $\boldsymbol{a}$; to prepare this step, the following expression for the Lorentz force can be derived from (A.6):

$$\frac{1}{4\pi}\boldsymbol{J}\times\boldsymbol{B} = -\frac{1}{4\pi R^2}[\nabla I^2/2 + (\Delta_\perp^*\Psi)\nabla\Psi] + (\Psi'/4\pi)\{\xi, I\}\nabla\zeta , \tag{A.22}$$

with the following definition of the operator $\Delta_\perp^*$:

$$\Delta_\perp^*\Psi = R^2\nabla_\perp\cdot\left(\frac{1}{R^2}\nabla_\perp\Psi\right), \tag{A.23 a}$$

$$= \frac{\partial^2\Psi}{\partial R^2} - \frac{1}{R}\frac{\partial\Psi}{\partial R} + \frac{\partial^2\Psi}{\partial z^2} . \tag{A.23 b}$$

In performing the scalar product of the Lorentz force with $\boldsymbol{a}$ we use (A.20); the first part of $\boldsymbol{a}$ which is parallel to $\boldsymbol{B}$ does not contribute, and the second part of $\boldsymbol{a}$ involving ω_0 gives only a contribution if it is multiplied with the last term in (A.22). Thus we have

$$\boldsymbol{a}\left(\frac{1}{4\pi}\boldsymbol{J}\times\boldsymbol{B}\right) = -(\varrho\omega_0\Psi'/4\pi)\cdot\{\xi, I\}$$

$$= -\varrho\Psi_0'\{\xi, \omega_0\pi_\zeta/\varrho\} , \tag{A.24}$$

where in the last step (A.16) has been used. Now we dot (A.14) with $\boldsymbol{a}$ and use the following rule valid for any function F which is independent of ζ:

$$\boldsymbol{a}\nabla F = \boldsymbol{a}_\perp\nabla F = (\nabla\zeta)[(\nabla\Psi_0)\times\nabla F] = \Psi_0'\{\xi, F\} .$$

In particular, the pressure term gives, according to (A.4, 5):

$$\boldsymbol{a}\nabla p = \Psi_0'\{\xi, \varrho T_0\} = \Psi_0'\varrho\{\xi, T_0 \ln\varrho\} . \tag{A.25}$$

The quantity $T_0 \ln\varrho$ is also the enthalpy of an ideal gas if it is compressed adiabatically with index $\gamma = 1$. Finally we use also the following rearrangement in (A.14):

$$\boldsymbol{a}\nabla(\varrho v^2/2) + (\boldsymbol{a}^2/2)\left(\boldsymbol{a}\nabla\frac{1}{\varrho}\right) = \varrho\left(\boldsymbol{a}\nabla\frac{\boldsymbol{a}^2}{2\varrho^2}\right) = \varrho\Psi_0'\{\xi, \boldsymbol{a}^2/2\varrho^2\} .$$

Collecting then all terms arising from (A.14) and dividing them by the common factor $\varrho\Psi_0'$ we have finally

$$\{\xi, U_0\} = 0 ,$$

with the following expression for the new (and arbitrary) flux function U_0:

$$U_0 = T_0 \ln\varrho + \Phi + \frac{\boldsymbol{a}^2}{2\varrho^2} + \omega_0\pi_\zeta/\varrho . \tag{A.26}$$

This is the "Bernoulli law" which generally can be used to determine π_ζ; in the particular case of (A.11) the Bernoulli law determines ϱ.

d) *Differential equation for* ξ

The last equation which has still to be fulfilled is the momentum balance equation (A.14) in the direction of $\nabla\xi$ which is orthogonal to both $\nabla\zeta$ and $\boldsymbol{a}$; it will result in a differential equation for ξ where the flux functions T_0, Ψ_0, Ψ, ω_0, I_0, and U_0 are given. Let us first generalize the prime of a flux function as introduced in (A.10) for any function F

as follows:

$$F' := \frac{\nabla\xi}{(\nabla\xi)^2}\nabla F\,. \tag{A.27}$$

Scalar multiplication of (A.14) with $\nabla\xi/(\nabla\xi)^2$ gives then

$$P' - (\boldsymbol{a}^2/2)(1/\varrho)' + \frac{\nabla\xi}{(\nabla\xi)^2}\left[\boldsymbol{\omega}\times\boldsymbol{a}/\varrho - \frac{1}{4\pi}\boldsymbol{J}\times\boldsymbol{B}\right] = 0 \tag{A.28}$$

with

$$P = p + \boldsymbol{a}^2/2\varrho\,. \tag{A.29}$$

According to (A.22) the contribution of the Lorentz force is

$$\frac{\nabla\xi}{(\nabla\xi)^2}\cdot\left(\frac{-1}{4\pi}\boldsymbol{J}\times\boldsymbol{B}\right) = \frac{1}{4\pi R^2}\left[\left(\frac{I^2}{2}\right)' + \Psi'\varDelta_\perp^*\Psi\right]. \tag{A.30}$$

With appropriate change of symbols ($4\pi\to-\varrho$; $I\to\pi_\zeta$; $\Psi\to\Psi_0$) we obtain also the contribution of $\boldsymbol{\omega}\times\boldsymbol{a}/\varrho$:

$$\frac{\nabla\xi}{(\nabla\xi)^2}\cdot(\boldsymbol{\omega}\times\boldsymbol{a}/\varrho) = -\frac{1}{\varrho R^2}\left[\left(\frac{\pi_\zeta^2}{2}\right)' + \Psi_0'\varDelta_\perp^*\Psi_0\right]. \tag{A.31}$$

The last term can be evaluated according to (A.23 a):

$$\varDelta_\perp^*\Psi_0 = R^2\nabla_\perp\left(\frac{1}{R^2}\Psi_0'\nabla_\perp\xi\right) = \Psi_0'\varDelta_\perp^*\xi + \Psi_0''(\nabla\xi)^2\,, \tag{A.32}$$

and a similar formula for the last term in (A.30). Defining now the quantities

$$Q = \frac{1}{2}\left(\frac{I^2}{4\pi} - \frac{\pi_\zeta^2}{\varrho}\right), \tag{A.33}$$

$$S = \frac{\Psi'^2}{4\pi} - \frac{\Psi_0'^2}{\varrho}\,, \tag{A.34}$$

and remembering that

$$\boldsymbol{a}^2 = (\Psi_0'^2/R^2)(\nabla\xi)^2 + \pi_\zeta^2/R^2\,, \tag{A.35}$$

we finally find from (A.28) the following differential equation for ξ: (A.36)

$$R^2(P' + \varrho\Phi') + Q' + (S/2)'(\nabla\xi)^2 + S\varDelta_\perp^*\xi = 0\,.$$

This is the desired equation for ξ with the expressions for P, Q, and S according to (A.29, 33, 34). The quantities ϱ, I, and π_ζ have to be expressed by the arbitrary flux functions according to a)−c), and by $(\nabla\xi)^2$ itself which enters through (A.35).